APPLYING BINOMIAL TREE TO PRICE MORTGAGES, CALIBRATE IMPLIED PREPAYMENT-ADJUSTED SPREAD (PAS), OPTIMIZE MORTGAGE REFINANCING, AND BUILD PAS INDEX

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ABSTRACT

This paper introduces an implied mortgage prepayment-adjusted spread (PAS) as a key measure for mortgagors to identify the richness or cheapness of mortgages, by applying the methodology of interest rate binomial trees based on benchmark yield curves such as treasury yield curve. Given the market accepted PAS, the optimal mortgage refinancing can be evaluated by measurements of mortgage refinancing efficiency, and the fair values for mortgage as well as prepayment (option) can be also calibrated. Finally the paper shows the mortgage PAS index could be established as a benchmark to gauge the mortgage rate premium charged by the market, and the PAS index could have potential applications in the financial world.

Key Words:

Mortgage Prepayment-Adjusted Spread (PAS); Implied PAS, Interest Rate Binomial Tree; Interest Rate Volatility; Mortgage Prepayment-Adjusted Spread Index (PAS Index).

JEF Classifiers: C32, C53

I. INTRODUCTION

Currently US Mortgage volume is over 10 trillion, one of the largest sectors in financial markets. The most common types of mortgages are 30-Year & 15-Year FRMs, and 5/1 ARMs. For example, 15-year 6.5% FRM can be interpreted as 6.5% interest rate fixed for 15 years. What does this mortgage term really mean? People may want to know what is implied by this term, and how do the lender charge for the spread (a static difference used in PV calculation comparing with benchmark yields such as treasury bonds) to cover credit risk, prepayment risk, market risks. Indeed, at the mortgagor's side, a mortgage can be viewed as a portfolio consisting of a short position in an optionless amortized risky bond and a long position in call (prepayment) option, so the fair value for the mortgage should be evaluated based on fair values for both positions (optionless amortized bond and prepayment option value). Since in US it is the most common that the prepayment option can be exercised without penalty at any time after the initial mortgage financed, this prepayment option is a kind of American Style. So it is quite hard to evaluate fair values for the mortgage, since there is no a perfect model to do so.

However, the methodology of the interest rate binomial tree can help to solve the puzzle. In order to make it work, we have to divide the horizon into a number of periods (such as yearly or monthly) to construct the binomial tree, and at each period there are a number of nodes (see more description in Section II). Like evaluating a plain vanilla American option by using a binomial tree method, the more periods used, the more accuracy could be achieved. But the interest rate binomial tree method is more complicated than the binomial one used for the plain vanilla American option, because we have to properly estimate the interest volatilities for each period, and then decompose the benchmark yield curve by trial and error methods (such as Bisection Method) since there is no analytical formula we can use. Thanks to modern computer technologies, we could solve the trees by given all necessary parameters.

In this paper, we use this methodology to construct interest rate binomial tree method, serving as a measurement or ruler, in trying to evaluate the mortgage value and its components (both the optionless amortized bond and the prepayment option value). For simplicity purpose, we just construct a yearly-based binomial tree (each period stands for one year time). With this method, we can calculate the implied prepayment-adjusted spread (PAS), a key measurement for the market charged premium for a mortgage. For example, based on the treasury par yield curve (as of 11/20/2008) and our estimated volatilities together with other assumptions, 15 year 6.5% FRM has 128 bps for implied prepayment-adjusted spread, and 236 bps for implied no-prepayment-adjusted (optionless) spread.

When we know the market accepted prepayment-adjusted spread (PAS) or option-adjusted spread (OAS), by using this binomial tree method we can calculate fair values of both existing or new mortgage, in evaluating mortgage optimal refinancing efficiency and identifying the required mortgage interest rate for 100% refinancing efficiency (the necessary condition for refinancing). The details are illustrated in Section III.

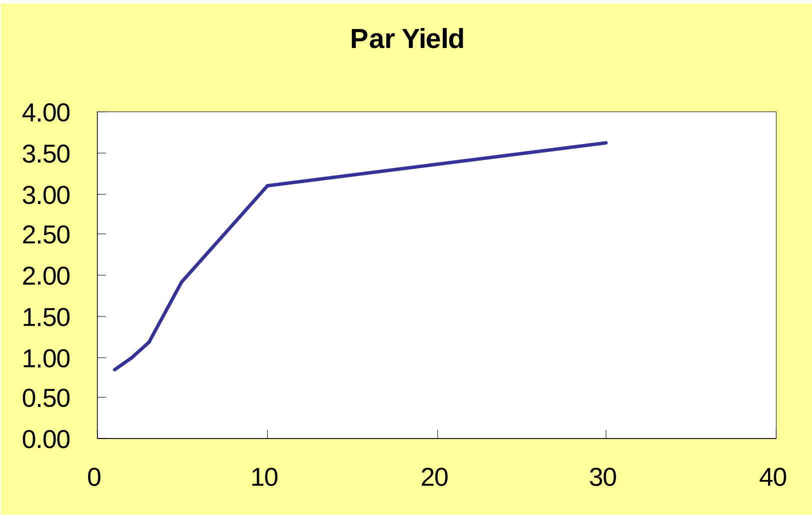
Finally, we explore an idea for building a PAS (Mortgage Prepayment-Adjusted Spread) Index, to gauge the average market charged premium for the mortgage loans in time series and to indicate hidden changes in the market. The PAS index might have broader applications in financial world, as it could be used for hedging or speculation purposes.

II. IMPLIED MORTGAGE PREPAYMENT-ADJUSTED SPREAD (PAS)

II.1 Benchmark Yield Curve and Binomial Tree

In order to gauge the richness or cheapness of a mortgage, at the first step we have to look into a benchmark rate (most commonly treasury rates). The treasury bonds are deemed as risk-free bonds because of they are guaranteed by the government, and with different maturities the yields, or par yield (the yield when its price is at par) for the treasury bonds are different. Usually the shorter maturities, the lower par yield, and the longer the higher. For example, the treasury par yields (at 11/20/2008) were 0.85% (1-year), 0.98% (2-year), 1.18% (3-year), 1.91% (5-year), 3.09% (10-year), and 3.63% (30-year) according to Bloomberg.com (see Figure 1).

Figure 1: Par Yield Curve (11/20/2008)

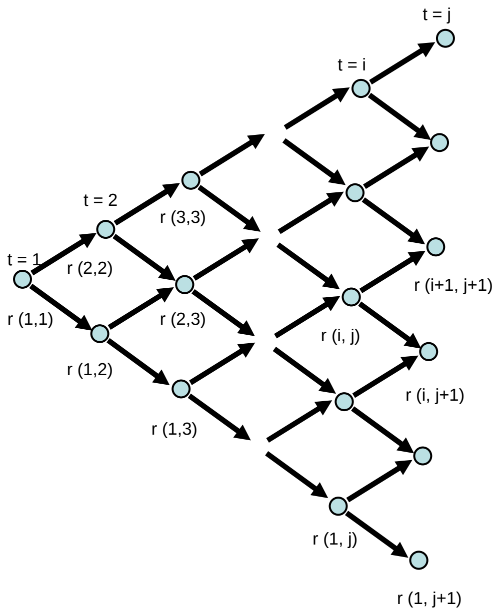


Using this benchmark curve, we could construct an interest rate binomial tree. Before doing so, we have to use these key rates to interpolate par yields for all maturities from 1 to 30 years by using a linear or non-linear interpolation method. For simplicity, we use the linear method to interpolate the par yields for the whole period of 30 years, which can be used to build an interest rate binomial tree. The structure of the binomial tree looks like a lattice (see Figure 2).

We start with the first node (t = 1), at which the interest rate is par yield (Year 1), then assume that the forward rates for Year 2 only have two possibilities either going down or up, to r(1, 2) or r(2,2) respectively. In this manner, each node only has two possibilities (either go down or up), so the tree can be extended to 30 periods to Year 30. We know that starting from Year 2, the forward rate at each node will be equal to the forward rate at its immediate next lower node (if exist) multiplied by an multiplier, $\exp(2\sigma_t)$. Here σ_t is the interest rate volatility at Year t.

So in general, $r(i+1, j) = r(i, j) \times \exp(2\sigma_j)$ (j = 2, 3, …30)

Figure 2: Binomial Tree



However, there are no analytical formula to calculate forward rates at all nodes (except for t =1) located the lower boundary line, i.e. r(1,2), r(1,3), r(1,4), …., r(1,30). But we could use some algorithms such as bisection or solver to find r(1,j) (j = 2, 3, ….30) by trial and error method starting from r(1,2) since par yield (2-year) is already known and its PV is at par with its coupon set to its par yield, assuming the two possibilities (either go down or up) of being equally likely. Once the r(1,2) is found, and then it is easily to calculate r(2,2), which equals

$r(2,2) = r(1,1) \times \exp(2\sigma_2)$, given σ_2 known.

Continuing to work in this way, we could find all values of r(1,j) (j = 2, 3, …, 30) in the lower boundary line, and then will be able to calculate all forward rates at all nodes in the binomial tree.

In this case, as we know the treasury par yield curve (30 years) as of 11/20/2008 using a linear interpolation method, if we assume the interest rate volatility constant at 10% (annually), the binomial tree can be constructed (see Binomial Tree in Appendix).

II.2 Introduce Implied (Market-Observed) Mortgage Prepayment-Adjusted Spread (PAS)

Once the interest rate binomial tree obtained, it will serve as a measurement or ruler to calibrate fair values for the mortgage values and its components. In reality, mortgages are callable amortized bonds, for the mortgagor (the borrower) issues an amortized straight bond (short an optionless amortized bond) with callable option (long a call option) callable at any time after mortgage financed without penalties (most cases in US).

Let’s assume that the mortgage can be only called at beginning of each year (like Bermudan Options) at a Replacement Value (RV) either through refinancing or simply paying-down, and RV is defined as follows.

$$RV_t = \max\left[(1+c)B_t, c_0\right]$$

Here

c: transaction cost (%) of outstanding balance;

c_0: minimum charge of the transaction ($) such as legal fee;

B_t: Outstanding balance of principal at the beginning of Year t, calculated as

$$B_t = B\frac{\left[1-(1+R)^{t-1-n}\right]}{\left[1-(1+R)^{-n}\right]} \quad \text{(t = 1, 2, 3, …, 30)}$$

B: is the initial mortgage amount.

To evaluate such an amortized callable bond (mortgage), we can use the ruler (binomial tree) to evaluate its cash flows at each nodes from the last period backward to the beginning node by discounted the forward rate at that node. At each node (t =1, 2, 3, …, 29), there are two cash flows from its next periods equally weighted plus the mortgage payment in that period, in comparison with the replacement value (RV) at that node because of optionality of prepayment. The formula is defined as below:

When t = 30,

PVs of cash flows (CF) at the beginning period of t = 30 is:

$$CF(i,30) = \min\left(\frac{PMT}{(1+r(i,30)/100+PAS/10000)}, RV_{30}\right) = \min\left(\frac{PMT}{(1+r(i,30)/100+PAS/10000)}, \max\left[(1+c)B_{30}, c_0\right]\right)$$

(i = 1, 2, 3, …, 30)

Here

PMT: yearly mortgage payment, which is fixed determined by initial mortgage value (B), fixed interest rate (R), and amortization period (n), i.e.

$$PMT = \frac{B\times R}{\left[1-(1+R)^{-n}\right]};$$

r(i,30): the i^{th} node in the binomial tree (t = 30);

PAS: Prepayment-Adjusted Spread (bps), which is a static spread (constant) for the entire tree;

RV_{30}: the replacement value at the beginning of t = 30, which is defined by the previous formula:

$$RV_{30} = \max\left[(1+c)B_{30}, c_0\right]$$

(B_{30} is the mortgage principal balance at the beginning of t = 30)

When t = 29, 28, ….,2, 1,

PVs of cash flows at node (i,j), the beginning period of t = j is:

$$CF(i,j) = \min\left(\frac{0.5\left[CF(i,j+1) + CF(i+1,j+1)\right] + PMT}{(1 + r(i,j)/100 + PAS/10000)}, RV_j \right)$$
$$= \min\left(\frac{0.5\left[CF(i,j+1) + CF(i+1,j+1)\right] + PMT}{(1 + r(i,j)/100 + PAS/10000)}, \max\left[(1+c)B_j, c_0\right] \right)$$

(j = 29, 28, …., 2, 1; i = 1, 2, …, j)

Here

CF(i, j) is the PV at the node (i ,j), the beginning of t = j;

CF(i, j+1) and CF(i+1,j+1) are PVs at the nodes (i, j+1) and (i+1, j+1), the beginning of t = j+1, respectively;

PMT: yearly mortgage payment;

r(i,j):

the i^{th} node in the binomial tree (t = j);

PAS: Prepayment-Adjusted Spread (bps), a static spread;

RV_j : the replacement value at the beginning of t = j, again defined by:

$$RV_j = \max\left[(1+c)B_j, c_0\right];$$

(B_j is the mortgage principal balance at the beginning of t = j)

The above PV formula also assumes that at each node the mortgagor (the borrower) will exercise her call option (make prepayment) when the present value is higher than the replacement value (RV), because she just need the value of RV to pay down the mortgage instead of paying the PV value through the future cash outflows, which is higher.

Now equipped with above tools or methodology, we will be able to calculate implied mortgage prepayment-adjusted spread (PAS), a key measure for identify the richness or cheapness of the mortgage.

The most common mortgages are Fixed Rate Mortgages (FRMs) and Adjusted-Rate Mortgages (ARMs) such as 30-year FRMs, 15-year FRMs, 5/1 ARMs, etc. Rates for FRMs are fixed for the entire life of the mortgage such as 30 or 15 years, while the rate for ARMs are fixed only for a short period such as 5 years, then rates will be adjusted at each year (usually linked to a benchmark rate plus premium), for example,

i) 30-Year 6% FRM (rate fixed at 6% for 30 years);

ii) 15-Year 5.5% FRM (rate fixed at 5.5% for 15 years);

iii) 30-Year 7% Jumbo (rate fixed at 7% for 30 years for large amount mortgage called "Jmbo");

iv) 5-Year 6.5% Jumbo (rate fixed at 6.5% for 15 years for Jumbo mortgage);

v) 5/1 5.8% ARM (rate fixed only for the first 5 years, after that, adjusted at each year referring to a benchmark rate plus a fixed premium);

vi) 5/1 6% ARM Jumbo (rate fixed only for the first 5 years, after that, adjusted at each year referring to a benchmark rate plus a fixed premium).

Next, we will calculate implied prepayment-adjusted spreads for some real mortgages observed in the current market as of 11/20/2008. The same treasury par yield curve in II.1 (as of 11/20/2008) is used, and the interest rate volatility is kept constant at different scenarios (10%, 20%, 30%, 40%, 50%, 60%, 70%, 80%). The transaction cost is 1% of the outstanding mortgage balance with minimum of $1000. The 5 chosen mortgages available in the market (11/20/2008) are as follows[1]:

	Lenders	**30 Yrs FRMs**
1	Bank of America	7.5%
2	Countrywide	7.375%
3	Citigroup	6.375%
4	National Mrtg Alliance	5.875%
5	QuickLoan	6.75%

Given the volatility constant at 20%, 30 Yrs 5.875% has an implied prepayment-adjusted spread of 193 bps, and 30 Yrs 7.5% has 371 bps. When the volatility rises to 50%, their implied PASs drop to 67 bps and 253 bps respectively. Clearly, when the volatility increases, the implied PAS decreases for all of different fixed mortgage rates, because the value of long position of the prepayment (call) option value increases with the rise of the volatility,, while the value of the short position (optionless amortized bond) almost unchanged, so the total value increases and in turn the PAS decreases. The details are shown in Table 1 and Figure 3.

Table 1: Implied PAS

Volatility	**7.500%**	**7.375%**	**6.750%**	**6.375%**	**5.875%**
10%	404	391	325	283	232
20%	371	357	288	247	193
30%	326	312	242	200	144
40%	288	273	202	160	103
50%	253	239	167	124	67
60%	225	210	138	95	37
70%	202	188	115	72	14
80%	184	169	97	54	-4

If treating mortgages as optionless (no prepayments) amortized bonds, we re-calculate implied spreads and the results shows that they are almost flat curves (Figure 4). 30 Yrs 5.875% FRM has an implied spread of about 240 bps (comparable to implied PAS at less than 10% of volatility level), and 30 Yrs 7.5% FRM has about 408 bps (comparable to implied PAS at less than 10% of volatility level). At the volatility of 10% level, the differences between the prepayment-adjusted and no-prepayment-adjusted spreads are very small. With increasing of the volatility, the differences become bigger, since the values of the prepayment option values increase. The flat curve of no-prepayment-adjusted spreads also indicates that the values of optionless amortized bonds are insensitive to the changes of volatilities.

[1] Source: www.bankrate.com, as of 11/20/2008

Figure 3: Implied PAS (30 Yr FRMs)

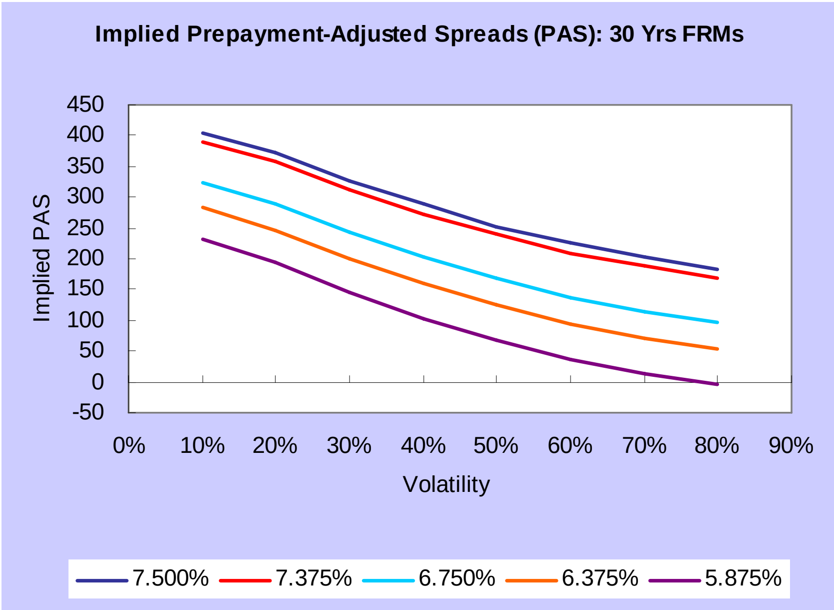
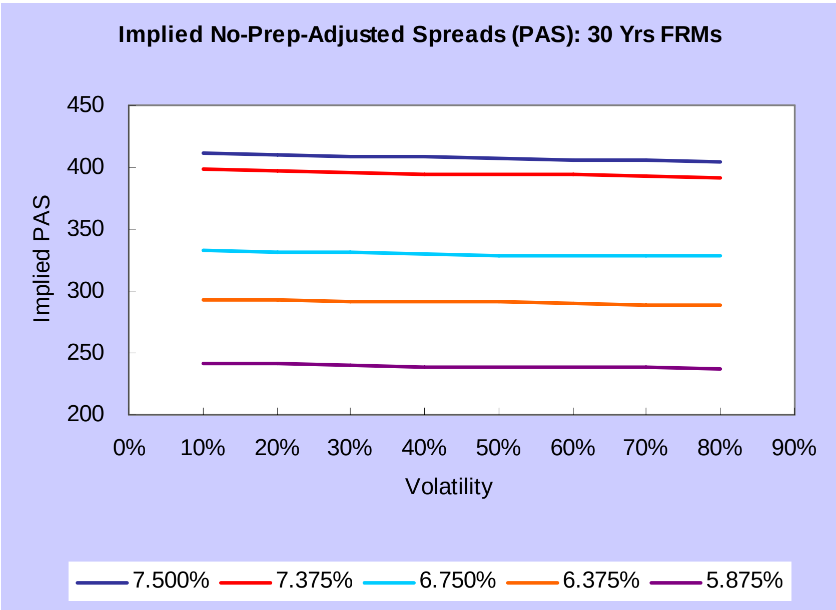


Figure 4: Implied Spreads (no prepayments)



Now we may wonder how about relatively short FRMs (such as 15 years fixed rate mortgages). Let's take the following examples[2]:

	Lenders	15 Yrs FRMs
1	QuickLoan	6.750%
2	Citigroup	6.125%
3	Mrtg Capital	5.545%
4	AmeriSave	4.625%

[2] Source: www.bankrate.com (as of 11/20/2008)

The implied prepayment-adjusted spreads for (15-Year FRMs) are given in Table 2, and graph shown in Figure 5.

Table 2: Implied PAS (15 Yrs FRMs)

Volatility	**6.750%**	**6.125%**	**5.545%**	**4.625%**
10%	393	328	269	174
20%	383	318	258	163
30%	357	292	231	134
40%	329	263	202	105
50%	299	233	171	73
60%	274	207	145	47
70%	252	185	123	25
80%	235	168	106	7

Similarly, when we treat 15Yrs FRMs mortgages as optionless (no prepayment) amortized bonds, re-calculate their spreads. They are also flat curves, for example, 15-Yr 4.625% FRM has about flat 174 bps, and 15-Yr 6.75% FRM about 391 bps, as shown in Figure 6. When the volatility is 10%, the prepayment-adjusted spreads has the same as no-prepayment spreads for all above mortgages, which means that at this volatility level, no-prepayments are expected to be exercised at fair value-judged approaches.

Figure 5: Implied PAS (15 Yrs FRMs)

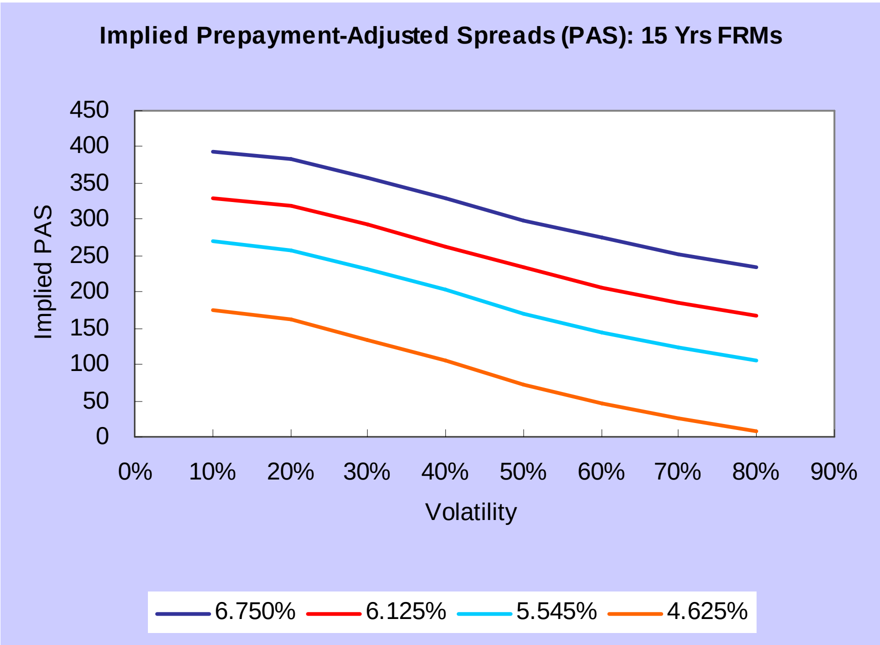
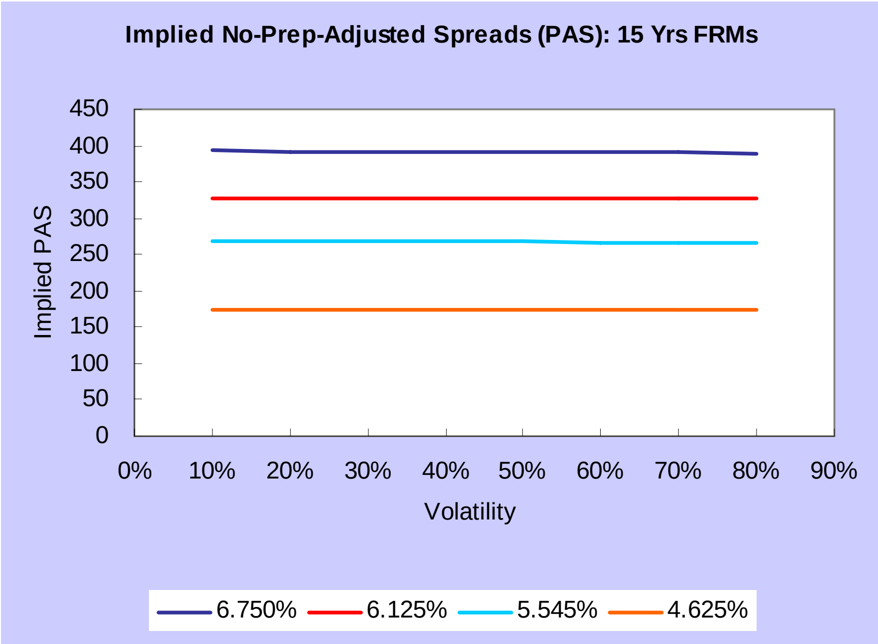


Figure 6: Implied No-Prepayment-Adjusted Spreads (15 Yrs FRMs)



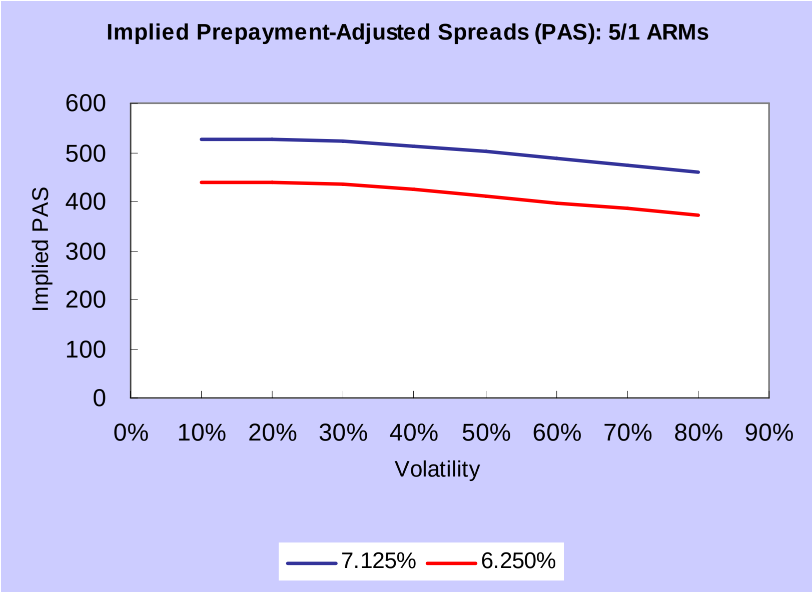
Finally, we think about ARMs (Adjusted Rate Mortgages) such as 5/1 7.125 ARMs (Countrywide), 5/1 6.25 ARMs (Citigroup)[3]. How to calculate their implied PASs? It is challenging, because we have to estimate the reference rates for the horizon after the locked period expires, and the binomial tree may not work well (Monte Carlo may be helpful). To make it simple without loss of much senses, we could treat ARMs as FRMs (for the locked period), and then after the end of the locked period the mortgage principal balance is considered to be paid down (anyway), since they become floating amortized bonds, which will lose sensitivities in duration measurements (The durations for floating notes are almost zero). In this way, we could use the same method for FRMs to calibrate their implied PAS. The results are given in Table 3 and graphs shown in Figure 7.

Table 3: Implied PAS (5/1 ARMs)

Volatility	7.125%	6.250%
10%	527	439
20%	527	438
30%	524	435
40%	514	425
50%	501	412
60%	488	398
70%	475	386
80%	461	371

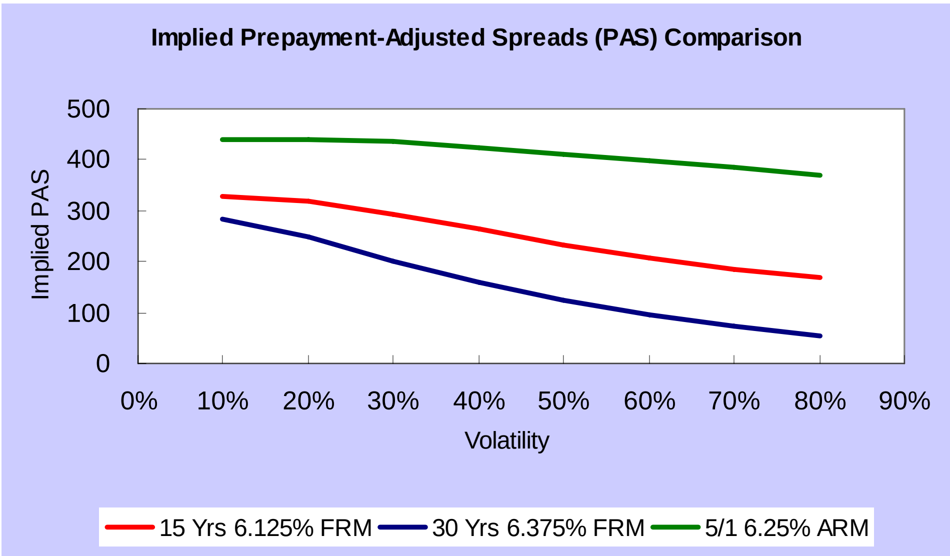
[3] Source: www.bankrate.com (as of 11/20/2008)

Figure 7: Implied PAS (5/1 ARMs)



In comparison with similar interest rates for 30 Yrs 6.375 % FRM, 15 Yrs 6.125% FRM and 5/1 6.25% ARMs, the result is graphed in Figure 8.

Figure 8: Comparison (30 Yrs & 15 Yrs FRMs, and 5/1 ARMs)



The above figures show that for similar interest rate, the longer the horizon mortgage, the lower the implied spread. Therefore, with similar interest rates, the longer horizon mortgage should be preferred, as it has lower implied spread (cheapness).

III. OPTIMAL MORTGAGE REFINANCING

The mortgage prepayment is a very complex phenomenon, which is hard to model because of many factors affecting homeowners to make decisions for prepaying their mortgage such as interest rate decreases, moves, job changes, etc.. Intuitively, we use the constructed binomial trees described in the previous section to evaluate the fair present values (PV), and option values for both existing mortgages and new mortgage with different terms, in trying to analyze mortgage refinancing efficiency. The refinancing efficiency is defined as a ratio of the change of PVs of cash flows to the change of option (prepayment) values, i.e.

$$Efficiency = \frac{\Delta PV\ of\ Cash\ Flows - Transaction\ Cost}{\Delta Option\ (\Pr epayment)\ Value}$$

For example, suppose that you have existing mortgage at fixed rate of 7% with remaining years of 25, you want to refinance it with a new 30-year fixed 5% mortgage because of drop in interest rates, is it worth to do so? For simplicity, we disregard tax-effects, but consider transaction costs. In order to answer this question, we can use our binomial tree with the exact method described in the previous section to calculate PVs and embedded option (prepayment) values for both the existing mortgage and the new one, given the known binomial tree, interest rate volatility, and market accepted spread (OAS or PAS in this paper). The prepayment option value for a mortgage can be defined as the PV difference between optionless (assumed no prepayment) amortized bond (mortgage) and callable (prepayment) amortized bond (mortgage). The same bond without optionality will have more value than with call option, so the option always has a non-negative value.

Now let's take the above example to calculate PV and option value of an existing mortgage (7% fixed remaining of 25 years), assuming

i) the interest rate volatility is constant at 15%;
ii) the market observed OAS or PAS (Prepayment Adjusted Spread) is 300 bps;
iii) the remaining principal balance is $200,000;
iv) the yield curve use as of 11/20/2008 released by US Treasury Department;
v) Transaction cost is $2,000 (1% of the principal balance);
vi) the prepayment only occurs at the beginning of each following year (the current time is the beginning of the year 6.

Based on our model, its PV and option value are as follows:

	Optionless	Prepayment Adjusted	Option Value
Fair Value	$211,904	$202,000	$9,904

Now considering to prepay the mortgage by refinancing with a new mortgage (30 Yr 6.25% FRM), as the interest rate goes down, we should calculate the PVs and its option value for the new mortgage. Assuming all settings are same, we calculate the values as below:

	Optionless	Prepayment Adjusted	Option Value
Fair Value	$196,060	$193,666	$2,394

So the refinancing efficiency is

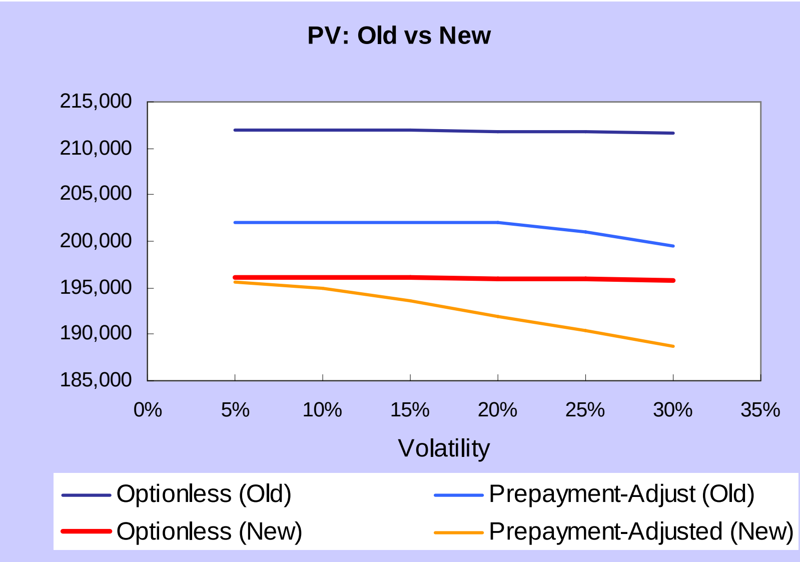
$$\frac{(202,000-193,666)-2,000}{9,904-2,394}=84\%$$

Because the efficiency is 84%, less than 100%, it should be considered as not optimal. So what interest rate of the new mortgage is required to achieve 100% refinancing efficiency? Based on our calculation, the rate should be less than 6.16% in order to reach 100% refinancing efficiency.

Now let's change the volatility from 15% to 5%, 10%, 20%, 25%, 30%, and all of others hold same, how is the refinancing efficiency, PVs of cash flows, and option values affected?

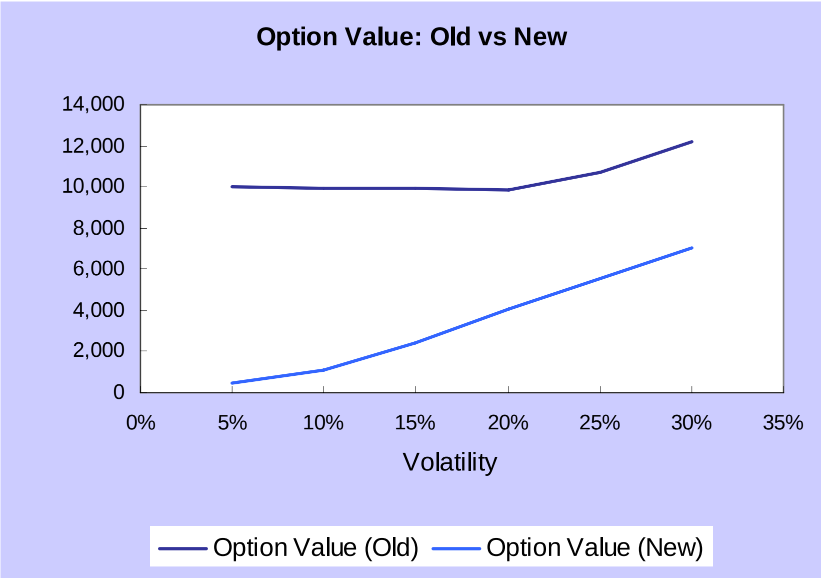
If it is optionless (no prepayment), PVs for both existing mortgage and new mortgage are very stable (almost flat curve) when the volatility changes. This reaffirms the result obtained in section II that optionless amortized bonds are insensitive to the volatility changes. But for callable (prepayment) mortgages, their PVs decrease when the volatility increase. The reason is that the mortgage (at mortgagee's side) is a portfolio of long position of an optionless amortized bond and short position of call (prepayment) option, and the value of optionless component has almost no changed with changes of the volatility, while the value for the short position decreases because the option value increases when the volatility rises. The result is shown in Figure 8.

Figure 8: PVs of Cash Flow (Old vs New Mortgage)



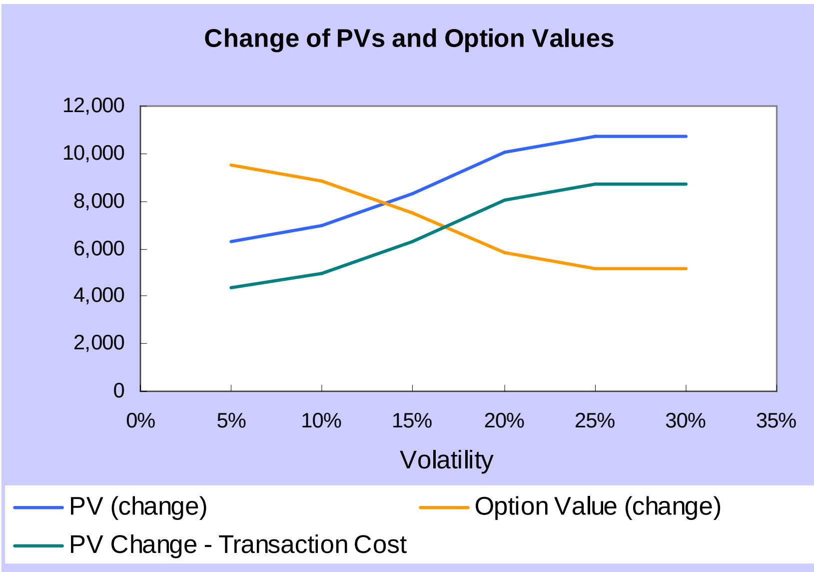
The option (prepayment) values for both the existing mortgage (old) and new mortgage are increase when the volatility increases. But the option value for the existing mortgage increases slowly than that of the new mortgage, because the new one has a long maturity (30 years) and lower coupon (interest rate at 6.25%) compared with the existing mortgage (25 years of remaining and 7% interest rate), which also indicates the new mortgage have longer duration that the old one. Consequently, the new mortgage is higher sensitive to the volatility change for the embedded option values (see Figure 9)

Figure 9: Option Value Comparison



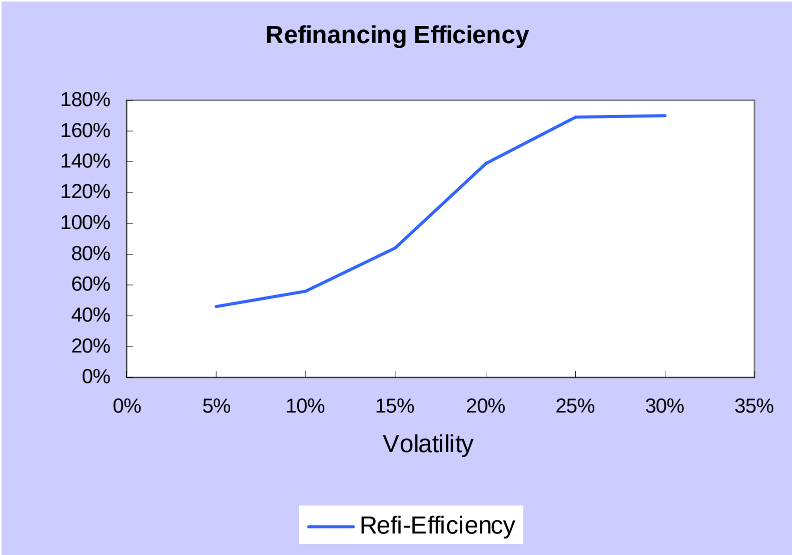
Like the option value, the refinancing efficiency is increasing when the volatility increases. This is because with volatility increase, the change of the option values (denominator in the refinancing efficiency formula) of the old and new mortgage shrinks (the value for the new mortgage increases faster than that of the old one), while the change of the prepayment-adjusted values (numerator) for the old and new mortgage increases (see Figure 10). So they will meet each other at some point, where the 100% refinancing efficiency could be reached.

Figure 10: Change in PVs and Option Values



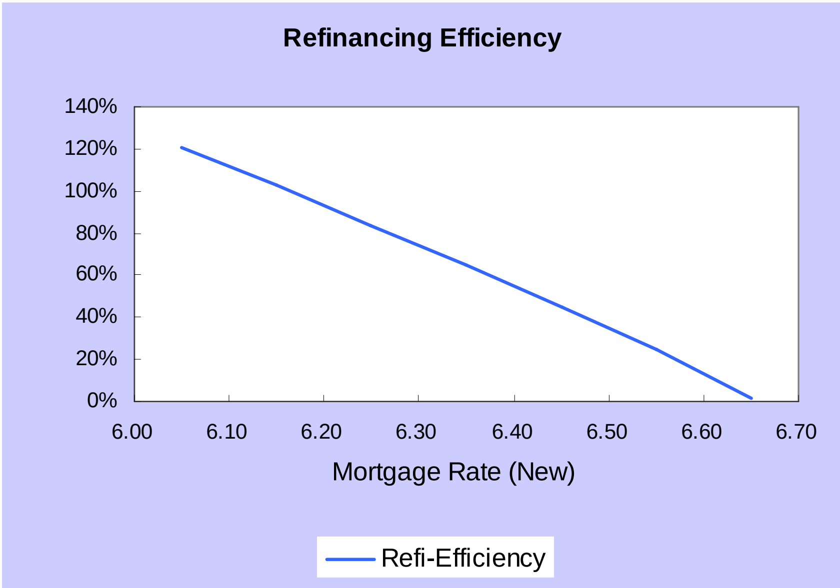
Therefore, the refinancing efficiency will increase with rising of the volatility (see Figure 11). In this case, when the volatility increases to about 18%, the refinancing efficiency will reach 100%, which means that it is worth to retire the existing mortgage (7% 25 years remaining FRM) through refinancing by the new mortgage (6.25% 30 years FRM) at the scenario of the volatility of 18%.

Figure 11: Refinancing Efficiency



When all others hold same, we change the new mortgage rate from 6.05% to 6.65%, and the refinancing efficiency changes from over 100% (at 6.15) to 2% (at 6.65),which means that the required maximum interest rate is 6.16% in order to achieve 100% refinancing efficiency (see Figure 12).

Figure 12: Refinancing Efficiency



IV. BUILD PAS INDEX

Mortgage rates offered by lenders are quite different, because they are affected by many factors such as interest rates, benchmark yields, business strategies, market competition, economic environments, etc.. Lenders also offer different mortgage rates to different borrowers usually based their creditworthiness, PTI ratio (Payment to Income Ratio), LTV ratio (Loan-to-Value Ratio). The implied prepayment-adjusted spreads (PAS) as well as the fair mortgage value and prepayment option value can be calculated by our model described in Section II, given the following known information and assumptions:

i) The benchmark (such as treasury) yield curves;

ii) Interest rate volatilities for all maturities, which can be estimated by using historical volatilities of benchmark yield curves, or using simulation technology such as Monte Carlo.

iii) Transaction cost;

iv) The offered mortgage rates with specifications;

v) The borrower is assumed to prepay her mortgage immediately when at the certain point (node) the PV of future cash out flow is greater than the RV (replacement value);

vi) The prepayment can be only made at the beginning of each period (node) for simplicity purpose, and when more nodes are included in the binomial tree, the more accuracy could be achieved, like using the binomial tree to price American options since American options can be exercised at any time during the life of the option. This also works for evaluating mortgage prepayment option value, as the mortgage can also be prepaid at any time during the life of mortgage. When more nodes added to the interest rate binomial tree, the structure is the same and the same methodology can be used to construct it. Although it is time-consuming, it will work thanks to the modern computer technologies.

Considering the national average mortgage rate for the most common types of mortgages (such as 30-Year FRMs, 15-Year FRMs, 5/1 ARMs) available at www.bankrate.com and GSEs (such as Freddie Mac), we could use our model based on the principles of consistency and continuity, to calculate the implied prepayment-adjusted spreads (PAS) for the national average mortgage rates in a timely manner. This PAS time series could be thought as PAS index, to gauge the market averaged accepted premium charged by the mortgage lenders. The PAS index can also considered as the market reaction to the interest rate changes, benchmark yield curve shifts, market tightness or ease of credits, market liquidities, economic conditions, future expectations. In the normal market conditions, when interest rate increases, the risk-free bond prices tend to decrease and consequently the yields increase, while the mortgage rates also tend to increase. In this case, the prepayment-adjusted spread may be increase or decrease somehow, dependent on what effects of tradeoffs between these two factors (yield curves shift up, and mortgage rates also up) as well as on some other factors such as the interest rate volatilities, transaction costs, tax-effects, etc. When the interest rate goes down, the risk-free bond prices increase and the yields decrease, but the mortgage rates also tend to decrease, and the prepayment-adjusted spread may fluctuate. However, when the market tends to be abnormal, the PAS index could jump up or down, which may indicate something really happening in the market, and it becomes a meaningful signal for the financial world including regulators, lenders, borrowers, financial practitioners, etc. During financial crisis, a flight to quality occurs, leading to increase prices of the risk-free bonds and significantly drive the treasury yields down, and treasury yield volatilities dramatically increase, while credits worsen and liquidity tightens and the mortgage rates increase, and consequently the PAS index should increase or jump significantly.

Now let's backtest the PAS index, and in order to do so, we have to collect all necessary data for our model including the following:

i) Historical treasury par yield curves (data during 1990/01/02 – 2008/11/20 available at US Treasury Department);

ii) Weekly national average mortgage rates for 30 Year FRM, and 15 Year FRM from 2007 and 2008 (up to 2008/11/20), released by Freddie Mac (see Table: National Average Mortgage Rate in Appendix);

iii) Volatilities set to the average of 1-year moving annualized treasury par yield volatility for all maturity (1 year – 30 years) after interpolations, and the annualized yield volatility is calculated as the daily standard deviation of the rate of return[4] scaled by $\sqrt{252} = 15.875$ (see historical par yield volatilities in Figure 14);

iv) Using $200,000 as initial mortgage loan, and transaction cost as 1% of the principal mortgage balance with minimum of $1,000;

Other assumptions include:

i) The borrower is assumed to prepay her mortgage immediately when at the certain point (node) the PV of future cash out flow is greater than the RV (replacement value);

ii) The prepayment can be only made at the beginning of each period or node (1 year) for simplicity purpose.

Based on our model (described in this paper), the PAS Index is calculated as follows in Figure 13 (Prepayment-Adjusted Spreads) & 14 (Non-Prepayment-Adjusted Spreads). See details data in Table: National Average Mortgage Rate and PAS Index in Appendix.

[4] $r_t = \ln(R_t/R_{t-1})$

Figure 13: Weekly PAS Index (bps): Prepayment-Adjusted Spreads

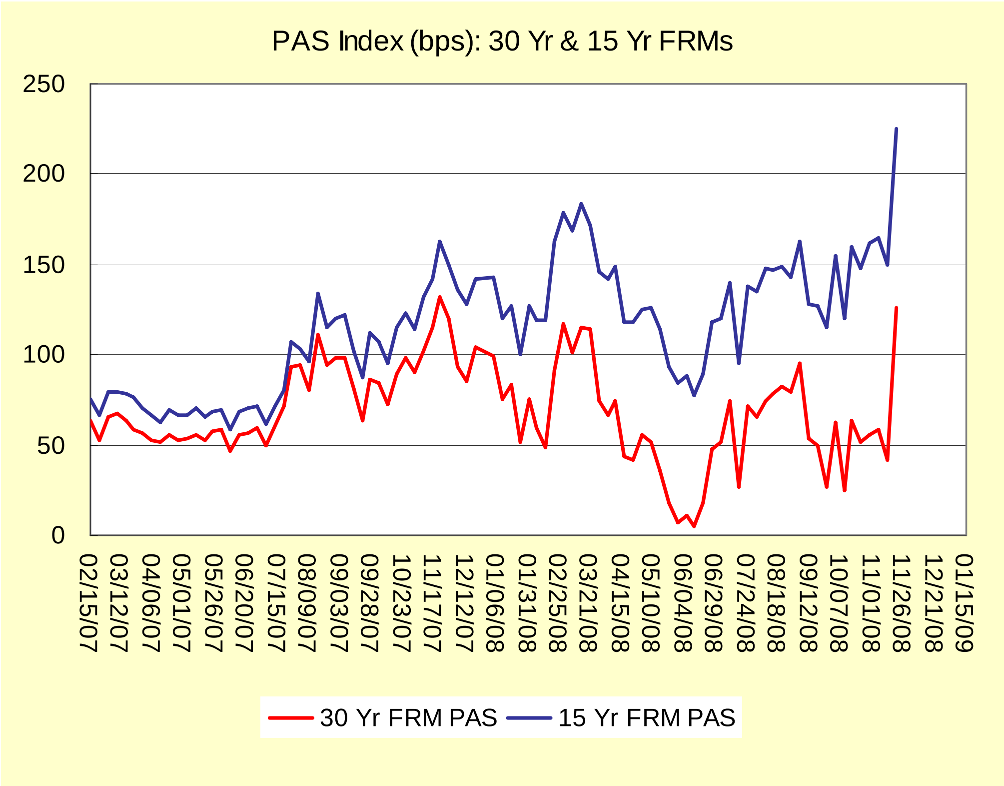
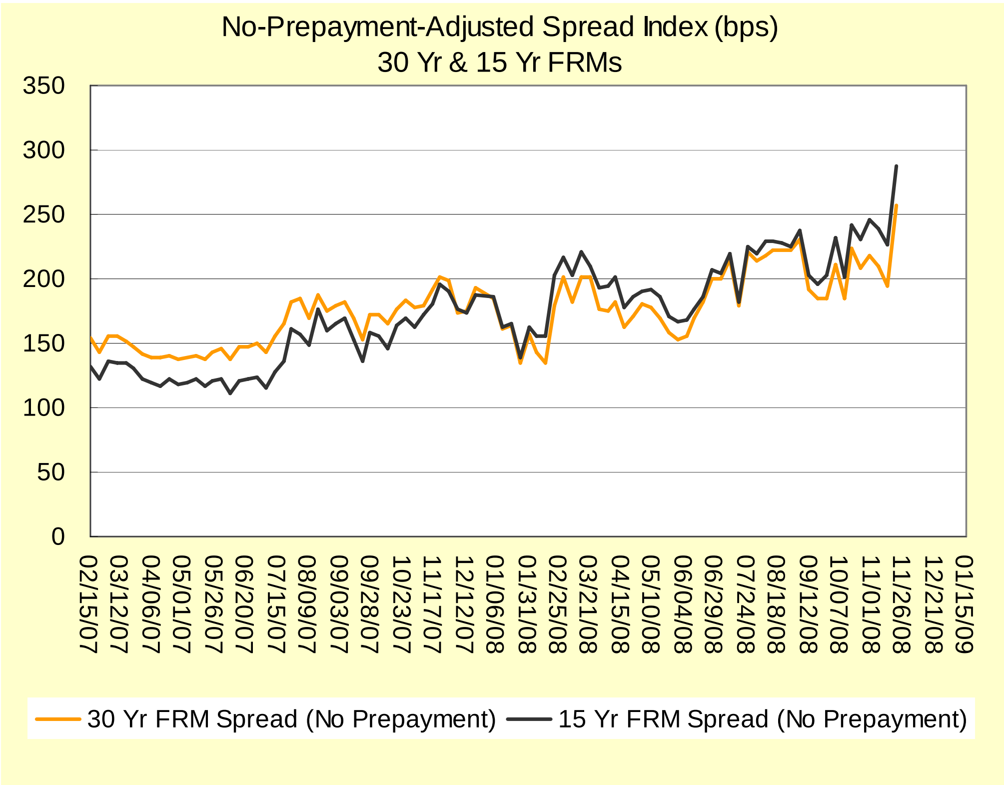


Figure 14: Weekly Spreads Index (bps): Non-Prepayment-Adjusted Spreads



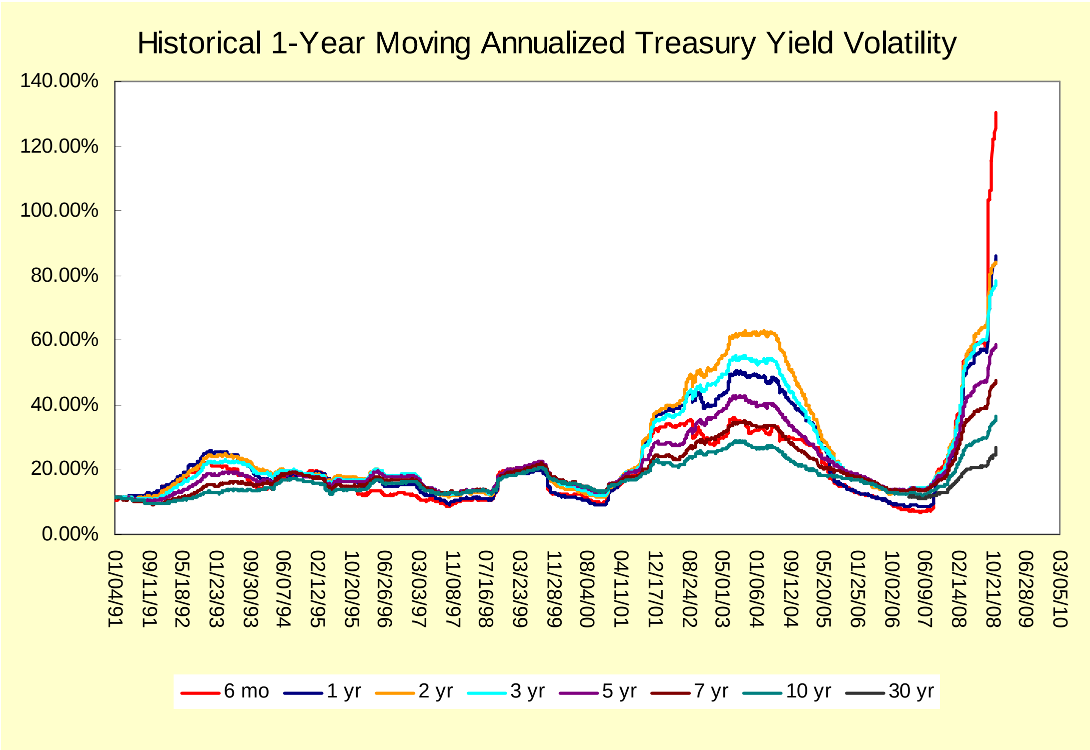
The weekly Prepayment-Adjusted Spread (PAS) Index for the period of 02/15/2007 to 11/20/2008 shows that the average PAS index for 30-year FRMs is 69 bps, about 47 bps lower than the average index of 15-year FRMs, which is 115 bps. This can be explained by that the longer maturity option will has higher option values (all other else hold same), which are not sufficiently offset by the mortgage rate difference between 30 year and 15 year FRMs (usually fixed mortgage rates for longer maturities are higher than for lower maturities). So the national average mortgage rate of 30 year FRMs is relatively cheaper than of 15 year FRMs.

The paths of both 30-year & 15-year FRM PAS indices show that during the first half of year 2007, the indices were relatively flat, at the lower level (about 57 bps for 30-year FRMs and 70 bps for 15-year bps), and during the last half of the year 2007, the PAS indices and their volatilities experienced significantly increasing as the subprime mortgage crisis emerging. But by the end of year 2007, the PAS index of 30-year FRMs dropped back to 50 bps, the similar level of the beginning of year 2007, and the PAS index of 15-year FRMs also retreated to about 100 bps, but about 25 bps higher than at the beginning of year 2007. During the first half of 2008, both indices were very volatile, peaked to 115 bps (30-year FRMs) and 183 bps (15-year FRMs) in the middle of March 2008, probably because of worsening of the subprime crisis and Bears Stearn collapse, and pulled back to the lowest level (almost zero spread for 30-year FRMs and 78 bps for 15-year FRMs in June 2008, as the government intervention and bailout for Bears Stearn. Then the indices dramatically were increasing during the third quarter of 2008 until the middle of September, as the subprime crisis became credit crisis and global financial crisis, and pulled back again to the relatively lower level and hovered over in October, as the congress passed the rescue plan for financial industries. In November 2008, however, both PAS indices sharply increased to 126 bps for 30-year FRMs (almost the highest level during the past 2 years) and to 225 bps for 15-year FRMs (nearly the highest level in the past 2 years), because worries for the government bailout failure soared and credits & liquidities were still tightening, and the world economy is entering a deep recession.

The average of no-prepayment-adjusted spreads for both 30-year and 15-year FRMs during the past 2 years were quite close each other, at about 175 bps. During 2007, the non-prepayment spreads for 30-year FRMs were about 20 bps higher than for 15-year FRMs, while during 2008, the average of no-prepayment-adjusted for 30-year FRMs were about 15 bps lower than for 15-year FRMs. They were also experienced the similar pattern as the prepayment-adjusted spread (PAS) indices, but less volatile, because of the prepayment option values were not included, which were significantly increasing during year 2008, because of the volatilities during 2008 were tremendously increasing, as shown in Figure 14.

During the period of the beginning of this year to 11/20/2008, the volatilities (1-year moving annualized par yield volatilities) for the key rates have been double or triple such as 1 year treasury par yield increasing from 24.9% to 85.7% (3.44 times), 2-year’s from 28.7% to 84.1% (2.9 times), 3-year’s from 27.9% to 78.1% (2.8 times), 5-year’s from 25.1% to 58.5% (2.3 times), 10-year’s from 18.9% to 36.5% (1.9 times) and 30-year’s from 15.4% to 26.7% (1.7 times). It is typically abnormal market changes during the financial crisis as the government intervened the market by tremendously cutting the interest rate, and investors still flew to quality (Safe Heaven) by purchasing treasury bonds.

Figure 14: 1-Year Moving Historical Treasury Par Yield Volatilities[5]



Now let' s look at the correlations between PAS indices, mortgage rates, treasury yields, stock indices during the same period, and the correlations are calculated as follows.

Correlation	30 Yr PAS	30 Yr NoP[6]	15 Yr PAS	15 Yr NoP	30 Yr Rate[7]	15 Yr Rate	10 Yr TY	30 Yr TY	Dow Index	SP500
30 Yr PAS	1.00	0.41	0.56	0.24	0.11	0.07	-0.12	-0.18	0.17	0.12
30 Yr NoP	0.41	1.00	0.84	0.93	0.15	0.06	-0.61	-0.66	-0.60	-0.64
15 Yr PAS	0.56	0.84	1.00	0.90	-0.25	-0.34	-0.81	-0.79	-0.53	-0.60
15 YrNoP	0.24	0.93	0.90	1.00	-0.13	-0.22	-0.81	-0.80	-0.72	-0.77
30 Yr Rate	0.11	0.15	-0.25	-0.13	1.00	0.97	0.66	0.62	0.20	0.23
15 Yr Rate	0.07	0.06	-0.34	-0.22	0.97	1.00	0.75	0.63	0.19	0.24
10 Yr TY	-0.12	-0.61	-0.81	-0.81	0.66	0.75	1.00	0.92	0.58	0.64
30 Yr TY	-0.18	-0.66	-0.79	-0.80	0.62	0.63	0.92	1.00	0.72	0.76
Dow Index	0.17	-0.60	-0.53	-0.72	0.20	0.19	0.58	0.72	1.00	0.99
S&P500	0.12	-0.64	-0.60	-0.77	0.23	0.24	0.64	0.76	0.99	1.00

[5] Data source: US Treasury Department (as of 11/20/2008)
The figures show that the short maturity treasury bonds have higher volatilities than the longer maturity bonds. Overall, 2 year treasury seems to have the highest volatilities, then comes 3 year and 1 year treasury bond. The 30 year treasury bond has the lowest volatilities. Most recently, the 6 month treasury bond experienced a sharp increase of its volatility.
[6] NoP: No-Prepayment-Adjusted Spread
[7] Rate: Average Mortgage Rate released by Freddie Mac.

From the figures of correlation coefficients, we noted that 30 Yr FRM PAS Index shows very low correlations (range between -0.18 to 0.17) with the average mortgage rates, 10-year & 30-year treasury yields, and Dow Index and SP 500, while 30-year FRM No-Prepayment-Adjusted Spread index has relatively high negative correlations with 10-year & 30-year treasury yields (-0.61 to -0.66) and Dow Index and SP 500 (about -0.6). For 15-year FRM indices, both PAS index and No-Prepayment-Adjusted Spread index show quite high negative correlations (about -0.80) with 10-year & 30-year treasury yields, and No-Prepayment-Adjusted index has about correlations (-0.72 to -0.77) with Dow Index and SP 500, while PAS index shows about -0.53 to -0.60 correlation with Dow and SP 500 indices.

Therefore, fund managers or financial practitioners could use PAS index to hedge their positions. For example, long/short position in equities could be hedged by long/short PAS index; long/short position in bonds may be possibly hedged by short/long PAS index, long/short interest rate volatilities may be hedged by long/short PAS index since when the volatilities go up PAS index tends to go down. Especially during market abnormal conditions, the PAS index may make more sense for hedge purposes.

V. CONCLUSION

The binomial tree method provides a relatively effective way in evaluating a mortgage values for both components (optionless amortized bond and prepayment option). As the prepayment option is an American style, which can be exercised at any time during life of the mortgage, it is required that the binomial tree has to be structured properly with enough periods to achieve accepted accuracy. In order to tradeoff between the accuracy and efficiency, it is recommended that the tree should be constructed on monthly based (i.e., each period stands for one month, and 12 periods in one year), which is also in line with the schedule of both mortgage payment (monthly) and prepayment. To do so, we could simply interpolate the benchmark yield curve on a monthly base and use monthly volatilities to apply the same methodology in building the binomial tree. Then we can assume that the prepayment can be only made at the beginning of each period, and at each node we use the same method described in this paper to calculate the PV of cash flows including consideration of the prepayment option, and then discount the cash flows at the forward rates generated from the corresponding benchmark yield plus a static prepayment-adjusted spread (PAS) backward to the first node to get the fair mortgage value. The prepayment option value is the optionless (no-prepayment) mortgage value minus the prepayment-adjusted mortgage value.

Once we have these values, the mortgage refinancing efficiency can be calculated. The mortgagor (borrower) can use the refinancing efficiency to optimize her refinancing efficiency, which requires a minimum of 100% of refinancing efficiency. Meanwhile, observing the market mortgage rates, we can calibrate their implied prepayment-adjusted spreads (PAS), a key measurement for the richness or cheapness of the mortgage.

By using such PAS concept, we could prospect to build a PAS index, which is proved to be feasible by using the technique introduced in this paper. This innovative index may have broader applications in financial worlds.

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VII. APPENDICES

VII.1 Binomial Tree (Sample truncated to Year 20)

																			T = 20
																		T = 19	27.04
																	T = 18	23.90	
																T = 17	21.14		22.14
															T = 16	18.71		19.56	
														T = 15	16.58		17.31		18.12
													T = 14	14.70		15.32		16.02	
												T = 13	13.04		13.57		14.17		14.84
											T = 12	11.57		12.03		12.54		13.11	
										T = 11	10.28		10.67		11.11		11.60		12.15
									T = 10	9.13		9.47		9.85		10.27		10.74	
								T = 9	13.69		8.42		8.74		9.10		9.50		9.95
							T = 8	11.05		7.48		7.76		8.07		8.41		8.79	
						T = 7	8.87		11.21		6.89		7.15		7.45		7.78		8.14
					T = 6	7.06		9.05		6.12		6.35		6.60		6.88		7.20	
				T = 5	5.54		7.26		9.17		5.64		5.86		6.10		6.37		6.67
			T = 4	5.08		5.78		7.41		5.01		5.20		5.41		5.64		5.89	
		T = 3	3.58		4.54		5.95		7.51		4.62		4.80		4.99		5.21		5.46
	T = 2	1.92		4.16		4.73		6.07		4.10		4.26		4.43		4.61		4.82	
T = 1	1.22		2.93		3.72		4.87		6.15		3.78		3.93		4.09		4.27		4.47
0.85		1.57		3.41		3.87		4.97		3.36		3.49		3.62		3.78		3.95	
	1.00		2.40		3.04		3.99		5.03		3.10		3.21		3.35		3.49		3.66
		1.29		2.79		3.17		4.07		2.75		2.85		2.97		3.09		3.23	
			1.96		2.49		3.26		4.12		2.53		2.63		2.74		2.86		3.00
				2.28		2.60		3.33		2.25		2.34		2.43		2.53		2.65	
					2.04		2.67		3.38		2.08		2.16		2.24		2.34		2.45
						2.13		2.73		1.84		1.91		1.99		2.07		2.17	
							2.19		2.76		1.70		1.76		1.84		1.92		2.01
								2.23		1.51		1.57		1.63		1.70		1.77	
									2.26		1.39		1.44		1.50		1.57		1.64
										1.24		1.28		1.33		1.39		1.45	
											1.14		1.18		1.23		1.29		1.35
												1.05		1.09		1.14		1.19	
													0.97		1.01		1.05		1.10
														0.89		0.93		0.97	
															0.83		0.86		0.90
																0.76		0.80	
																	0.71		0.74
																		0.65	
																			0.60

VII.2

Table: National Average Mortgage Rate and Prepayment-Adjusted Spread (PAS) & No-Prepayment-Adjusted Spreads Indices

	30 Yr FRM			15 Yr FRM		
Date	Mrtg Rate	Prep-Adj Index	No-Prep-Adj Index	Mrtg Rate	Prep-Adj Index	No-Prep-Adj Index
2007-2-15	6.30	63.3	153.7	6.03	75.4	131.7
2007-2-22	6.22	52.1	143.2	5.97	66.1	122.6
2007-3-1	6.18	65.6	155.5	5.92	79.8	135.8
2007-3-8	6.14	67.4	155.5	5.86	79.8	134.6
2007-3-15	6.14	63.1	152.0	5.88	78.7	134.1
2007-3-22	6.16	58.8	146.6	5.90	76.3	130.6
2007-3-29	6.16	56.3	142.1	5.86	70.2	122.8
2007-4-5	6.17	52.6	139.0	5.87	66.6	119.6
2007-4-12	6.22	51.5	139.0	5.90	62.6	116.2
2007-4-19	6.17	55.1	140.7	5.89	69.8	121.9
2007-4-26	6.16	52.9	137.5	5.87	66.8	118.5
2007-5-3	6.16	53.8	139.5	5.87	66.8	119.3
2007-5-10	6.15	55.9	140.4	5.87	70.5	122.1
2007-5-17	6.21	52.7	137.0	5.92	65.8	116.4
2007-5-24	6.37	57.5	143.6	6.06	68.6	120.3
2007-5-31	6.42	58.5	146.0	6.12	69.4	121.7
2007-6-7	6.53	46.9	137.2	6.22	58.8	111.7
2007-6-14	6.74	55.9	147.4	6.43	68.3	121.2
2007-6-21	6.69	56.9	146.9	6.37	70.0	122.5
2007-6-28	6.67	59.9	150.0	6.34	71.1	123.6
2007-7-5	6.63	49.7	143.1	6.30	61.3	115.9
2007-7-12	6.73	60.4	155.5	6.39	71.4	127.7
2007-7-19	6.73	71.2	165.1	6.38	80.0	135.8
2007-7-26	6.69	93.3	182.5	6.37	107.6	161.0
2007-8-2	6.68	94.2	184.3	6.32	103.1	157.3
2007-8-9	6.59	80.1	168.8	6.25	96.2	149.2
2007-8-16	6.62	111.0	187.4	6.30	133.9	176.2
2007-8-23	6.52	94.5	175.2	6.18	115.0	160.2
2007-8-30	6.45	97.8	178.9	6.12	119.9	165.9
2007-9-6	6.46	98.5	182.2	6.15	121.6	169.6
2007-9-13	6.31	81.7	169.8	5.97	102.5	152.7
2007-9-20	6.34	63.3	152.6	5.98	86.8	135.7
2007-9-27	6.42	86.5	172.6	6.09	111.9	158.6
2007-10-4	6.37	84.7	172.9	6.03	106.9	155.6
2007-10-11	6.40	72.7	164.7	6.06	95.1	146.4
2007-10-18	6.40	89.1	176.6	6.08	115.3	163.3
2007-10-25	6.33	98.5	183.3	5.99	123.2	169.6

2007-11-1	6.26	90.4	177.5	5.91	114.3	162.0
2007-11-8	6.24	102.0	178.9	5.90	132.0	172.1
2007-11-15	6.24	115.0	191.2	5.88	141.8	181.2
2007-11-21	6.20	132.3	200.9	5.83	162.4	196.2
2007-11-29	6.10	119.8	198.7	5.73	149.8	190.2
2007-12-6	5.96	93.4	173.8	5.65	135.5	176.3
2007-12-13	6.11	85.6	174.9	5.78	128.2	173.9
2007-12-20	6.14	103.8	192.5	5.79	142.0	188.0
08-1-3	6.07	98.9	185.4	5.68	142.8	186.8
08-1-10	5.87	75.0	160.6	5.43	119.9	162.8
08-1-17	5.69	83.8	164.0	5.21	126.6	165.7
08-1-24	5.48	51.7	134.4	4.95	100.6	138.4
08-1-31	5.68	75.8	156.9	5.17	127.4	162.8
08-2-7	5.67	59.6	142.7	5.15	118.7	155.0
08-2-14	5.72	49.0	134.3	5.25	118.7	156.0
08-2-21	6.04	90.8	178.6	5.64	162.7	202.4
08-2-28	6.24	117.4	201.6	5.72	178.8	216.7
08-3-6	6.03	101.0	182.4	5.47	168.9	203.1
08-3-13	6.13	115.4	201.5	5.60	183.1	220.7
08-3-20	5.87	114.2	201.9	5.27	171.6	209.1
08-3-27	5.85	74.7	176.7	5.34	146.0	193.3
08-4-3	5.88	66.7	175.2	5.42	141.8	194.0
08-4-10	5.88	74.9	182.5	5.42	149.3	200.8
08-4-17	5.88	43.3	162.3	5.40	118.0	177.4
08-4-24	6.03	41.8	170.3	5.62	118.2	185.5
08-5-1	6.06	55.2	181.2	5.59	124.6	190.9
08-5-8	6.05	51.3	177.5	5.60	126.5	191.8
08-5-15	6.01	36.1	169.6	5.60	113.6	186.3
08-5-22	5.98	18.2	158.2	5.55	93.3	171.1
08-5-29	6.08	7.1	153.4	5.66	84.6	166.4
08-6-5	6.09	11.2	155.7	5.65	88.3	168.6
08-6-12	6.32	4.6	169.9	5.93	77.7	176.1
08-6-19	6.42	18.1	181.3	6.02	89.7	186.6
08-6-26	6.45	47.5	200.4	6.04	118.0	206.5
08-7-3	6.35	51.4	199.6	5.92	120.0	204.1
08-7-10	6.37	74.7	214.6	5.91	140.3	218.8
08-7-17	6.26	27.0	179.2	5.78	95.0	181.6
08-7-24	6.63	71.3	221.3	6.18	138.1	224.9
08-7-31	6.52	65.4	213.3	6.07	134.5	219.3
08-8-7	6.52	74.4	218.6	6.10	147.7	229.7
08-8-14	6.52	77.9	221.9	6.07	147.2	229.2
08-8-21	6.47	81.9	222.9	6.00	148.8	228.1
08-8-28	6.40	79.1	222.4	5.93	142.8	224.4

08-9-4	6.35	95.2	230.1	5.90	163.1	237.8
08-9-11	5.93	53.5	191.6	5.54	127.8	202.4
08-9-18	5.78	50.1	184.5	5.35	127.2	196.0
08-9-25	6.09	26.9	185.3	5.77	114.8	203.2
08-10-2	6.10	62.6	210.6	5.78	154.6	232.0
08-10-9	5.94	24.5	184.7	5.63	120.2	201.8
08-10-16	6.46	63.0	223.6	6.14	159.7	241.4
08-10-23	6.04	51.6	208.7	5.72	148.1	230.2
08-10-30	6.46	55.6	218.6	6.19	161.9	245.5
08-11-6	6.20	58.7	209.6	5.88	164.9	239.6
08-11-13	6.14	41.5	194.5	5.81	149.7	226.1
08-11-20	6.04	125.8	256.5	5.73	224.8	287.0

Source:

Average mortgage rates (30-year & 15-year) released by Freddie Mac as of 11/20/2008